Your Signature ____________________

Instructions:

(a) Please write your name on every page.
(b) Maximum time is 3 hours. Please stop writing when you are asked to do so.
(c) You may use any result proved in class. When using a result, please state the result precisely.
(d) Do not use any results from homework assignments.
(e) Provide adequate justification for answers to the questions below.

Score

1.	(20)	
2.	(30)	
3.	(20)	
4.	(30)	
Total.	(100)	

Please attach this sheet to your answer script when you turn them in

1. Let $f : [-1, 1] \to \mathbb{R}$ be Riemann integrable on $[-1, 1]$ and continuous at 0. For any $h \in [-1, 1]$ let

$$I_h = \begin{cases} [0, h] & \text{if } 0 \leq h \\ [h, 0] & \text{if } h \leq 0 \end{cases}$$

Find the $\lim_{h \to 0} \int_{I_h} f$.

2. Let $\alpha \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{|x_1|^\alpha x_2}{x_1^2 + x_2^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) For which values of α, is f continuous at 0 ?
(b) Let $\alpha = 2$. Does f have all directional derivatives at 0 ?
(c) Let $\alpha = 2$. Is f differentiable at 0?

3. Decide whether each of the following statements are true or false providing adequate justification.

(a) The set

$$A = \{f : \mathbb{R} \to (0, 2] \mid f^{-1} \text{ exists and both } f, f^{-1} \text{are continuous}\}$$

is non-empty.

(b) Let $a < b$ be two real numbers and $f : [a, b] \to \mathbb{R}$ is a bounded function. If f^3 is Riemann integrable then f is Reimann integrable.

(c) $A = \mathbb{Q} \cup [0, 1]$ is a compact subset of $\mathbb{R}$.

4. Let $P = (0, \infty)$. Consider the function $d : P \times P \to [0, \infty)$ given by

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| \text{ for any } x, y \in P.$$

Show that:
(a) (P, d) is a metric space.
(b) Let $\epsilon > 0$. Describe the open ball centered at 1 with radius ϵ in P.
(c) Is the interval $(2, 3)$ open in (P, d) ?
(d) Is the metric space (P, d) complete ?